A solution to Haagerup's problem on normal weights

Ikhan Choi (崔 瀷瀚)

The University of Tokyo

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Main result

In my master's thesis, I solved a 50-year-old problem posed by Haagerup in his master's thesis and obtained the following theorem!

Theorem ([Cho25], arXiv:2501.16832)

Let A be a C^*-algebra, and F^* be a weakly* closed convex hereditary subset of A^{*+}. Then, for any $\omega \in A^{*+} \setminus F^*$, there exists $a \in A^+$ such that

$$\omega(a) > 1 \qquad \text{and} \qquad \omega'(a) \le 1 \qquad \text{for all } \omega' \in F^*.$$

The original proof has been simplified thanks to N. Ozawa. The positivity condition $a \ge 0$ is the non-trivial point. Applying the idea used to prove the above theorem, I could simplify the proof of the following.

Theorem ([Haa75])

For a subadditive weight φ on M, the followings are equivalent:

- *φ is σ-lower semi-continuous.*
- *φ is given by the pointwise supremum of normal positive linear functionals.*

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C*-algebras and von Neumann algebras

We will always denote a C*-algbera and a vN algebra by A and M respectively.

A vN (or W*-) algebra can be defined as a C*-algebra M that admits a predual M_*, which is unique if it exists. The canonical weak* topology on M is conventionally called the σ-weak topology, and normality usually means the σ-weak continuity. We focus on the dual pairs (A, A^*) and (M, M_*) and their weak/weak* topologies. Note that for a convex subset of A or M_* it is norm closed iff it is weakly closed by the Hahn-Banach separation.

To see the measure theoretic analogues in today's talk, the following notes would be helpful to keep in mind.

Example (Commutative C*-algebras)

A commutative A is $*$-isomorphic to $C_0(X)$ for a locally compact Hausdorff space X. The positive part of its dual A^{*+} is given by the set of finite regular Borel measures on X.

Example (Commutative vN algebras)

A commutative M is $*$-isomorphic to $L^\infty(X, \mu)$ for a localizable measure space (X, μ). The positive part of its predual M_*^+ is isomorphic to $L^1(X, \mu)$, which can be identified with the set of finite measures on X absolutely continuous with respect to μ. Note that every σ-finite measure is a localizable.

Definitions on weights

Definition (Weights and subadditive weights)

A *weight* on A is a homogeneous additive functional $\varphi : A^+ \to [0,\infty]$.
A *subadditive weight* on A is a homogeneous subadditive functional $\varphi : A^+ \to [0,\infty]$.

Definition (Properties of weights)

For a weight φ on A, we say it is

- (i) *faithful* if $\varphi(a) = 0$ implies $a = 0$ for $a \in A^+$,
- (ii) *densely defined* if $\varphi^{-1}([0,\infty))$ is norm dense in A^+,
- (iii) *lower semi-continuous* if $\varphi^{-1}([0,1])$ is norm closed in A^+.

For a weight φ on M, we say it is

- (ii') *semi-finite* if $\varphi^{-1}([0,\infty))$ is σ-weakly dense in M^+,
- (iii') *normal* if $\varphi^{-1}([0,1])$ is σ-weakly closed in M^+.

A positive linear functional gives rise to a weight. Normality can be regarded as the generalization of the countable additivity of measures. Note that the σ-weak lower semi-continuity can be understood as a restatement of the Fatou lemma.

Motivating examples for weights

Example (Localizable measures)

A localizable measure μ is always a faithful semi-finite normal weight on $L^\infty(\mu)$. In fact, every M admits a faithful semi-finite normal weight.

Example (Radon measures)

Densely defined lower semi-continuous weights on $C_0(X)$ for a locally compact Hausdorff X are exactly positive linear functionals on $C_c(X)$, and are exactly locally finite inner regular Borel measures on X. Every densely defined subadditive weight on a unital A is bounded, so there is A without a faithful densely defined lower semi-continuous weight.

Example (Gelfand-Naimark-Segal representations)

There is a one-to-one correspondence between weights on A and unitary equivalence classes of *semi-cyclic representations* of A, which is defined as a representation $\pi : A \to B(H)$ equipped with a partially defined left A-linear map $\Lambda : A \to H$ of dense range. The corresponding weight φ on M is normal iff π is normal and Λ is σ-weakly closed. A densely defined lower semi-continuous weight φ on A gives rise to a faithful semi-finite normal weight on the σ-weak closure A'' in the associated semi-cyclic representation to φ.

Equivalent characterizations for normality of weights

Theorem ([Haa75])

For a weight φ on M, the followings are all equivalent.

(1) *φ is completely additive for positive elements*;

$$\varphi(\sum_i x_i) = \sum_i \varphi(x_i), \qquad x_i \in M^+.$$

(2) *φ preserves directed suprema*;

$$\varphi(\sup_i x_i) = \sup_i \varphi(x_i), \qquad x_i \uparrow \sup_i x_i \text{ in } M^+.$$

(3) *φ is σ-weakly lower semi-continuous*;

$$\varphi(\lim_i x_i) \le \liminf_i \varphi(x_i), \qquad x_i \to \lim_i x_i \ \sigma\text{-weakly in } M^+.$$

(4) *φ is given by the pointwise supremum of normal positive linear functionals*;

$$\varphi(x) = \sup_{\omega \le \varphi,\ \omega \in M_*^+} \omega(x), \qquad x \in M^+.$$

(4)⇒(3)⇒(2)⇒(1) are clear.

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First problem

Problem (1.10)

For a subadditive weight on M, is it σ-weakly lower semi-continuous if it preserves directed suprema?

Haagerup proved (1)⇒(3) directly without an intermediate step (2). He proved first for σ-finite vN algebras, and extended the result to general vN algebras.

Definition

A vN algebra is called *σ-finite* or *countably decomposable* if every orthogonal family of non-zero projections is countable, or equivalently, it admits a faithful normal state.

Theorem ([Haa75])

For a weight on σ-finite M, (1)⇒(3) *holds.*
For a subadditive weight on general M, (1)⇒(3) *holds if every restriction on σ-finite vN subalgebras satisfies* (1)⇒(3).
Therefore, for a weight on general M, (1)⇒(3) *holds.*

Thus, it is enough to solve the problem in the σ-finite case.

Second problem

Problem (1.11)

For a weight on M, is it normal if it is normal on every commutative vN subalgebra?

Theorem ([Dix53])

For a positive linear functional ω on M, the followings are all equivalent.

(0) *ω is completely additive for orthogonal projections.*

(1) *ω is completely additive for positive elements.*

(2) *ω preserves directed suprema.*

(3) *ω is σ-weakly continuous.*

In particular, a (positive) linear functional on M is normal if it is normal on every commutative vN subalgebra by (0)⇒(1), and it is used to prove some equivalent characterizations for weak compactness in M_*.

Example

(0)⇒(1) is false for weights. Define a weight φ for $x = (x_n)_n \in \ell^\infty(\mathbb{N})^+$ such that $\varphi(x) := \sum_n x_n$ if $x \in c_c(\mathbb{N})$ and $\varphi(x) := \infty$ otherwise. Then, it gives a counterexample.

Third problem

Problem (2.7)

Does the positive bipolar theorem hold for dual C^*-algebras? See (d) *in the below.*

Definition

Let (E,E^*) be a dual pair of (directed partially) ordered real vector spaces such that E^+ and E^{*+} are mutually dual cones, i.e. $E^{*+} = \{x^* \in E : x^*(x) \geq 0 \text{ for } x \in E\}$. For $F \subset E^+$, we say it is *hereditary* if $0 \leq x \leq y \in F$ implies $x \in F$, and its *positive polar* is the set

$$F^{r+} := (F^r)^+ = \{x^* \in E^* : \sup_{x \in F} x^*(x) \leq 1\}^+ = \{x^* \in E^{*+} : \sup_{x \in F} x^*(x) \leq 1\}.$$

Theorem ((a)∼(c) in [Haa75], (d) in [Cho25])

Consider the ordered real dual pairs (M^{sa}, M_*^{sa}) and (A^{sa}, A^{sa*}) of self-adjoint parts.

(a) *If F is a σ-weakly closed convex hereditary subset of M^+, then $F = F^{r+r+}$.*

(b) *If F_* is a norm closed convex hereditary subset of M_*^+, then $F_* = F_*^{r+r+}$.*

(c) *If F is a norm closed convex hereditary subset of A^+, then $F = F^{r+r+}$.*

(d) ***If F^* is weakly* closed convex hereditary subset of A^{*+}, then $F^* = (F^*)^{r+r+}$.***

Corollaries of third problem

Corollary ([Haa75])

For a subadditive weight φ on M, (3)⇒(4) holds.

(3) *φ is σ-lower semi-continuous.*

(4) *φ is given by the pointwise supremum of normal positive linear functionals.*

Proof. Define

$$F := \{x \in M^+ : \varphi(x) \leq 1\}, \qquad F_* := \{\omega \in M_*^+ : \omega \leq \varphi\}.$$

Then, $F_* = F^{r+}$ by definition, and (4) is equivalent to $F = F_*^{r+}$. Since F is σ-weakly closed by the σ-weak lower semi-continuity of φ, and since F is clearly convex and hereditary by definition of subadditive weights, so we are done by the part (a). □

Using (c) instead of (a), we can simplify the proof of the following theorem.

Corollary ([Com68])

A lower semi-continuous subadditive weight on A is given by the pointwise supremum of positive linear functionals.

Corollaries of third problem

Corollary

There are one-to-one correspondences

$$\left\{\begin{array}{c}\textit{normal}\\ \textit{subadditive}\\ \textit{weights on } M\end{array}\right\} \longleftrightarrow \left\{\begin{array}{c}\sigma\textit{-weakly closed}\\ \textit{convex hereditary}\\ \textit{subsets of } M^+\end{array}\right\} \longleftrightarrow \left\{\begin{array}{c}\textit{norm closed}\\ \textit{convex hereditary}\\ \textit{subsets of } M_*^+\end{array}\right\}$$

and

$$\left\{\begin{array}{c}\textit{lower semi-}\\ \textit{continuous}\\ \textit{subadditive}\\ \textit{weights on } A\end{array}\right\} \longleftrightarrow \left\{\begin{array}{c}\textit{norm closed}\\ \textit{convex hereditary}\\ \textit{subsets of } A^+\end{array}\right\} \longleftrightarrow \left\{\begin{array}{c}\textit{weakly}^* \textit{ closed}\\ \textit{convex hereditary}\\ \textit{subsets of } A^{*+}\end{array}\right\}$$

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Proof of (a)

Definition (Suppression by the one-parameter family of functional calculi)

For $\delta > 0$, we define a function $f_\delta : (-\delta^{-1}, \infty) \to \mathbb{R}$ such that

$$f_\delta(t) := t(1+\delta t)^{-1}, \qquad t > -\delta^{-1}.$$

They are operator monotone, σ-strongly continuous, and has the semi-group property.

Proof sketch of (a) by Haagerup. Since

$$\begin{aligned} F^{r+} &= F^r \cap M_*^+ = F^r \cap (-M^+)^r = (F \cup -M^+)^r = (F - M^+)^r, \\ F^{r+r+} &= (F - M^+)^{rr+} = (\overline{F - M^+})^+ \end{aligned}$$

by the usual real bipolar theorem, it suffices to show $(\overline{F - M^+})^+ \subset F$.

Heuristics. Let $x \in (\overline{F - M^+})^+$ with nets x_i and y_i such that $x_i \to x$ σ-weakly in M and $x_i \le y_i \in F$ for all i. Observe that if x_i were bounded by $r > 0$, then assuming $x_i \to x$ σ-strongly and $f_\delta(y_i) \to y_\delta$ σ-weakly for each $0 < \delta < r^{-1}$, we get

$$x_i \le y_i \in F, \qquad f_\delta(x_i) \le f_\delta(y_i) \in F, \qquad 0 \le f_\delta(x) \le y_\delta \in F, \qquad f_\delta(x) \in F, \qquad x \in F.$$

The boundedness of x_i is neceesary to define $f_\delta(x_i)$ for δ independently of i.

Proof of (a)

Question. How can we remove the boundedness condition of x_i?

Solution. We use the Krein-Šmulian theorem. Define

$$G := \{x \in M^{sa} : \text{for any sufficiently small } \delta > 0,\ f_\delta(x) \in F - M^+\}.$$

It is enough to show

$$F - M^+ \subset G, \qquad G^+ \subset F, \qquad \overline{G} \subset G,$$

and the first two are clear. To apply the Krein-Šmulian theorem, fix $r > 0$ and let $M_r := \{x \in M : \|x\| \le r\}$. The proof of the σ-weak closedness is divided into the two steps: $G \cap M_r$ is σ-strongly closed, $G \cap M_r$ is convex. After proving these, $\overline{G} \subset G$ by the Krein-Šmulian theorem. □

Proof of (b)

Definition (Bounded commutant Radon-Nikodym derivatives)

Let $\psi \in M_*^+$ and let $\pi : M \to B(H)$ be associated cyclic representation to ψ with the canonical cyclic vector $\Omega \in H$. Then, we have a positive bounded linear map $\theta : \pi(M)' \to M_*$, which we call the *RN map*, defined such that

$$\theta(h)(x) := \langle h\pi(x)\Omega, \Omega\rangle, \qquad h \in \pi(M)',\ x \in M.$$

If $\omega \in M_*$ satisfies $|\omega(x)| \le \psi(x)$ for all $x \in M^+$, then $\theta^{-1}(\omega)$ is uniquely defined and $\|\theta^{-1}(\omega)\| \le 1$, which has $\theta^{-1}(\omega) = d\omega/d\psi$ when M is commutative.

Proof sketch of (b). It is enough to prove $(\overline{F_* - M_*^+})^+ \subset F_*$. Let $\omega \in (\overline{F_* - M_*^+})^+$ with sequences ω_n and φ_n such that $\omega_n \to \omega$ in norm of M_* and $\omega_n \le \varphi_n \in F_*$ for all n. We may assume $\|\omega_n - \omega\| \le 2^{-n}$ for all n by passing to a subsequence. Define

$$\psi := \omega + \sum_n [\omega_n - \omega] + \sum_n 2^{-n} \frac{\varphi_n}{1 + \|\varphi_n\|} \in M_*^+,$$

and let $\theta : \pi(M)' \to M_*$ be the RN map associated to ψ. Since $-\psi \le \omega_n \le \psi$ implies the boundedness $\|\theta^{-1}(\omega_n)\| \le 1$ for all n, the weak convergence $\omega_n \to \omega$ in M_* implies the convergence $\theta^{-1}(\omega_n) \to \theta^{-1}(\omega)$ in the weak operator topology of $\pi(M)'$.

By the Mazur lemma, we can take a net ω_i in the convex hull of ω_n such that $\theta^{-1}(\omega_i) \to \theta^{-1}(\omega)$ strongly in $\pi(M)'$, and the corresponding $\varphi_i \in F^*$ can be defined such that $\omega_i \leq \varphi_i$ for all i. (In fact, the net ω_i can be taken to be a sequence because the commutant is σ-finite by the existence of the separating vector, but it is not necessary in here.) For each i and $0 < \delta < 1$, define

$$\omega_\delta := \theta(f_\delta(\theta^{-1}(\omega))), \qquad \omega_{i,\delta} := \theta(f_\delta(\theta^{-1}(\omega_i))), \qquad \varphi_{i,\delta} := \theta(f_\delta(\theta^{-1}(\varphi_i))).$$

Then, the strong convergence $f_\delta(\theta^{-1}(\omega_i)) \to f_\delta(\theta^{-1}(\omega))$ in $\pi(M)'$ implies $\omega_{i,\delta} \to \omega_\delta$ weakly in M_*, and the strong convergence $f_\delta(\theta^{-1}(\omega)) \to \theta^{-1}(\omega)$ in $\pi(M)'$ implies $\omega_\delta \to \omega$ weakly in M_* as $\delta \to 0$. If we define $\varphi_\delta := \theta(\lim_i f_\delta(\theta^{-1}(\varphi_i)))$ by taking a subnet or a cofinal ultrafilter, then $\varphi_{i,\delta} \to \varphi_\delta$ weakly in M_*. Since $\omega_i \leq \varphi_i$ and $0 \leq \varphi_{i,\delta} \leq \varphi_i \in F_*$, we get

$$\omega_{i,\delta} \leq \varphi_{i,\delta} \in F_*, \qquad 0 \leq \omega_\delta \leq \varphi_\delta \in F_*, \qquad \omega_\delta \in F_*, \qquad \omega \in F_*.$$

This completes the proof. □

Strategies for (d)

Let ω_i and φ_i be nets in A^{*sa} such that $\omega_i \to \omega$ weakly* in A^* and $\omega_i \le \varphi_i \in F^*$ for all i.

Question 1. How can we choose the reference ψ for the Radon-Nikodym?

Solution 1. Take ψ_i dynamically depending on ω_i.

Question 2. How can we commute the weak* limit of ω_i and f_δ without strong topology?

Solution 2. Approximate f_δ with affine functions by

$$t - \delta^{\frac{1}{2}} \le f_\delta(t) \le t, \qquad |t| \le 2^{-1}\delta^{-\frac{1}{4}},$$

$$(1+\delta^{-1})t \le f_\delta(t) \le t, \qquad 0 \le t \le 1.$$

Proof of (d)

Proof of (*d*). It suffices to show $(\overline{F^* - A^{*+}})^+ \subset F^*$. Define

$$G^* := \left\{ \omega \in A^{*sa} : \begin{array}{c} \text{there is } \psi \in A^{*+}, \text{ and there is } \varphi_\delta \in F^* \\ \text{for any sufficiently small } \delta > 0, \text{ such that} \\ \|\psi\| \le 1,\ \|\varphi_\delta\| \le \delta^{-1}, \text{ and } \omega \le \varphi_\delta + \delta^{\frac{1}{2}}\psi \end{array} \right\}.$$

It suffices to show $F^* - A^{*+} \subset G^*$, $G^{*+} \subset F^*$, and $\overline{G^*} \subset G^*$.

Proof of (d)

Step 1. Let $\omega \in F^* - A^{*+}$. Take $\varphi \in F^*$ such that $\omega \le \varphi$. Define, for $\delta > 0$,

$$\psi := \frac{[\omega]}{1+\|\omega\|} + \frac{\varphi}{(1+\|\omega\|)(1+\|\varphi\|)}, \qquad \varphi_\delta := \theta(f_\delta(\theta^{-1}(\varphi))),$$

where θ is the RN map associated to ψ. The norm conditions $\|\psi\| \le 1$ and $\|\varphi_\delta\| \le \delta^{-1}$ are easily checked. For sufficiently small $\delta > 0$ such that $\|\theta^{-1}(\omega)\| \le 1 + \|\omega\| \le 2^{-1}\delta^{-\frac{1}{4}}$ and $\delta \le 1$, we have

$$\theta^{-1}(\omega) \le f_\delta(\theta^{-1}(\omega)) + \delta^{\frac{1}{2}} \le f_\delta(\theta^{-1}(\varphi)) + \delta^{\frac{1}{2}},$$

so $\omega \le \varphi_\delta + \delta^{\frac{1}{2}}\psi$ and $\omega \in G^*$.

Proof of (d)

Step 2. Let $\omega \in G^{*+}$. Take $\psi \in A^{*+}$ and $\varphi_\delta \in F^*$ such that $\|\psi\| \le 1$, $\|\varphi_\delta\| \le \delta^{-1}$, $\omega \le \varphi_\delta + \delta^{\frac{1}{2}}\psi$, for any sufficiently small $\delta > 0$. Let $\psi_\delta := \omega + \delta\varphi + \psi$, and let θ_δ be the associated RN map. For any fixed $\delta' > 0$, since $0 \le \theta_\delta^{-1}(\omega) \le 1$, we have

$$\begin{aligned}0 &\le (1+\delta')^{-1}\theta_\delta^{-1}(\omega) \le f_{\delta'}(\theta_\delta^{-1}(\omega)) \le f_{\delta'}(\theta_\delta^{-1}(\varphi_\delta + \delta^{\frac{1}{2}}\psi)) \\ &\le f_{\delta'}(\theta_\delta^{-1}(\varphi_\delta) + \delta^{\frac{1}{2}}) \le f_{\delta'}(\theta_\delta^{-1}(\varphi_\delta)) + \delta^{\frac{1}{2}},\end{aligned}$$

and it implies

$$0 \le (1+\delta')^{-1}\omega \le \theta_\delta(f_{\delta'}(\theta_\delta^{-1}(\varphi_\delta))) + \delta^{\frac{1}{2}}\psi_\delta.$$

Since $\|\psi_\delta\| \le \|\omega\| + 2$ is bounded and $\theta_\delta(f_{\delta'}(\theta_\delta^{-1}(\varphi_\delta))) \in F^*$ is also bounded for fixed δ' as $\delta \to 0$, by considering the limit along a cofinal ultrafilter on the set of δ, we have $(1+\delta')^{-1}\omega \in F^*$, so $\delta' \to 0$ gives $\omega \in F^*$.

Proof of (d)

Step 3. To show G^* is weakly* closed, we claim for any $r > 0$ that

$$\overline{(F^* - A^{*+}) \cap A^*_{2r}} \subset G^*, \qquad G^* \cap A^*_r \subset \overline{(F^* - A^{*+}) \cap A^*_{2r}},$$

where $A^*_r := \{\omega \in A^* : \|\omega\| \le r\}$. If these are true, then

$$G^* \cap A^*_r = \overline{(F^* - A^{*+}) \cap A^*_{2r}} \cap A^*_r$$

is weakly* closed and convex in A^* for all $r > 0$, so the Krein-Šmulian theorem shows the claim.

Proof of (d)

Let $\omega_i \in (F^* - A^{*+}) \cap A^*_{2r}$ be a net such that $\omega_i \to \omega$ weakly* in A^*. Following the proof of $F^* - A^{*+} \subset G^*$, we can take $\psi_i \in A^{*+}$ and $\varphi_{i,\delta} \in F^*$ such that $\|\psi_i\| \le 1$, $\|\varphi_{i,\delta}\| \le \delta^{-1}$, $\omega_i \le \varphi_{i,\delta} + \delta^{\frac{1}{2}}\psi_i$, for uniformly sufficiently small δ such that $1 + 2r \le 2^{-1}\delta^{-\frac{1}{4}}$ because $\|\omega_i\|$ is bounded by $2r$. Since the three conditions are preserved by the weak* convergence, taking the limit along a cofinal ultrafilter on the index set of i, we can obtain limit points ψ and φ_δ so that $\omega \in G^*$.

Let $\omega \in G^* \cap A^*_r$. Take $\psi \in A^{*+}$ and $\varphi_\delta \in F^*$ with $\|\psi\| \le 1$, $\|\varphi_\delta\| \le \delta^{-1}$, $\omega \le \varphi_\delta + \delta^{\frac{1}{2}}\psi$, for any sufficiently small $\delta > 0$. If $\delta^{\frac{1}{2}} < r$, then $\omega - \delta^{\frac{1}{2}}\psi \in (F^* - A^{*+}) \cap A^*_{2r}$ converges to ω weakly* in A^* as $\delta \to 0$, we have $\omega \in \overline{(F^* - A^{*+}) \cap A^*_{2r}}$. □

References I

[Cho25] Ikhan Choi. A solution to Haagerup's problem and positive Hahn-Banach separation theorems in operator algebras, 2025. arXiv 2501.16832.

[Com68] Fran cois Combes. Poids sur une C^*-algèbre. *J. Math. Pures Appl. (9)*, 47:57–100, 1968.

[Dix53] J. Dixmier. Formes linéaires sur un anneau d'opérateurs. *Bull. Soc. Math. France*, 81:9–39, 1953.

[Haa75] Uffe Haagerup. Normal weights on W^*-algebras. *J. Functional Analysis*, 19:302–317, 1975.