KK-theory as a stable infinity category

Ikhan Choi

The University of Tokyo

Matsuyama, September 2025

Contents

Simplicial sets

Definition (Simplicial sets)

A *simplicial set* is a functor $X : \Delta^{\mathrm{op}} \to \mathrm{Set}$, where Δ is the category of non-empty finite totally ordered sets with morphisms as monotone functions. We write $X_n := X([n])$.

Since Δ has a skeleton $\{[n] : n \geq 0\}$ with $[n] := \{0, \cdots n\}$, a simplicial set X is a diagram

$$\cdots \qquad X_2 \;\substack{\xrightarrow{\;d_0^2\;} \\ \xleftarrow{\;s_1^1\;} \\ \xrightarrow{\;d_1^2\;} \\ \xleftarrow{\;s_0^1\;} \\ \xrightarrow{\;d_2^2\;}}\; X_1 \;\substack{\xrightarrow{\;d_0^1\;} \\ \xleftarrow{\;s_0^0\;} \\ \xrightarrow{\;d_1^1\;}}\; X_0$$

where the face d and degeneracy s satisfy some relations called the simplicial identities.

Example (Standard simplices and horns)

For $0 \leq i \leq n$, simplicial sets Δ^n and Λ^n_i are defined such that

$$\Delta^n_m := \mathrm{Hom}_\Delta([m],[n]), \qquad \Lambda^n_{i,m} := \{\alpha \in \Delta^n_m : \alpha([m]) \not\supset [n] \setminus \{i\}\}.$$

By the Yoneda, element of X_n is identified to a simplicial map $\Delta^n \to X$.

Infinity categories

Definition

We say a simplicial set Y satisfies the *Kan condition*satisfies the *Segal condition*is an (small) ∞*-category* if for all $0 \leq^{\forall} i \leq^{\forall} n 0 <^{\forall} i <^{\forall} n$

$$\begin{array}{ccc} \Lambda^n_i & \longrightarrow & Y \\ \downarrow & \nearrow_{\exists !!!!} & \\ \Delta^n & & \end{array} \qquad \text{(horn filling conditions)}$$

Example (Singular simplicial sets)

A simplicial set satisfies the Kan condition iff it is "homotopic to" $\mathrm{Sing}(T)$ defined such that $\mathrm{Sing}(T)_n := C(|\Delta^n|, T)$ for a topological space T.

Example (Nerves)

A simplicial set satisfies the Segal condition iff it is "isomorphic to" $N(\mathcal{C})$ defined such that $N(\mathcal{C})_n := \mathrm{Fun}([n], \mathcal{C})$ for a small category $\mathcal{C}$, and $N : \mathrm{Cat} \to \mathrm{Cat}_\infty$ is fully faithful.

Composition and homotopy

Let $\mathcal{C}$ be an ∞-category.

- Suppose $f : \Delta^1 \to \mathcal{C}$ and $g : \Delta^1 \to \mathcal{C}$ satisfy
$$x = d_0 f, \qquad d_1 f = y = d_0 g, \qquad d_1 g = z.$$
This composable pair defines $(g, f) : \Lambda^2_1 \to \mathcal{C}$. Then, it has an extension $\sigma : \Delta^2 \to \mathcal{C}$. We say $h := d_1\sigma$ is a *composition* of $f = d_2\sigma$ and $g = d_0\sigma$, but it is not unique.
$$\begin{array}{ccc} & y & \\ {}^{f}\nearrow & & \searrow^{g} \\ x & & z \end{array} \quad \rightsquigarrow \quad \begin{array}{ccc} & y & \\ {}^{f}\nearrow & \exists\sigma & \searrow^{g} \\ x & \xrightarrow[h]{} & z \end{array}$$

- Suppose $f : \Delta^1 \to \mathcal{C}$ and $g : \Delta^1 \to \mathcal{C}$ satisfy
$$d_0 f = x = d_0 g, \qquad d_1 f = y = d_1 g.$$
If there is $\sigma : \Delta^2 \to \mathcal{C}$ s.t. $d\sigma = (s_0 y, g, f)$ or $(f, g, s_0 x)$ or $(s_0 y, f, g)$ or $(g, f, s_0 x)$, equivalent by the horn filling for $n = 3$, then we say f and g are *homotopic*.
$$\begin{array}{ccc} & y & \\ {}^{f}\nearrow & \exists\sigma & \searrow^{s_0 y} \\ x & \xrightarrow[g]{} & y \end{array} \quad \Longleftrightarrow \quad \begin{array}{ccc} & x & \\ {}^{s_0 x}\nearrow & \exists\sigma & \searrow^{f} \\ x & \xrightarrow[g]{} & y \end{array} \quad \Longleftrightarrow \quad \text{(two more)}$$

Dwyer-Kan localization

Many examples of infinity categories are constructed by a notion of localization.

Definition

For a relative category $(\mathcal{C}, \mathcal{W})$, there is a universal functor $\mathcal{C} \to L_{\mathcal{W}}\mathcal{C}$ to an ∞-category that sends every morphism in $\mathcal{W}$ to a (homotopy) equivalence.
It is called the (Dwyer-Kan) *localization* of $\mathcal{C}$ at $\mathcal{W}$ in the sense of ∞-categories.

The conventional construction is divided into two steps: simplicial localization, and homotopy coherent nerve with fibrant replacement Ex^∞.

The term 'Dwyer-Kan localization' originally referred only to the first step, the simplicial localization, but since Hinich ([Hin16]) it seems to have also come to mean the entire process of an ∞-categorical localization.

Bunke, Land, or the other researchers on KK-theory also follow the term 'Dwyer-Kan localization'.

Stable infinity categories

Definition

Recall that an Ab-enriched category is called *abelian* if it is pointed, admits (co)kernels, and every mono/epi is a kernel/cokernel respectively.

In some analogy, an infinity category is called *stable* if it is pointed, admits (co)kernels, and every morphism is a kernel and is a cokernel.

A gap on their third conditions is related to the existence of a "suspension functor".
A "suspension functor" is a kind of auto-functor, a result of "stabilization".

Example

- An abelian category $\mathcal{A}$ can be "stabilized" to the stable ∞-category $\mathcal{D}(\mathcal{A})$. One way to obtain this is the DK localization of $\mathrm{Ch}(\mathcal{A})$ at quasi-isomorphisms.
- Pointed spaces Top_* can be "stabilized" to the stable ∞-category Sp. One way to obtain this is to consider Ω-spectra.

Abelianness of a category is an additional data, we need to specify a Ab-enrichment.
Stability of an ∞-category is a property, and it induces a natural Sp-enrichment.

Contents

Classical KK-theory

By Higson, we can define the KK-theory as follows.

Definition ([Hig87], [Hig88])

The *KK-theory* is the universal functor $\mathrm{C}^*\mathrm{Alg}_{\mathrm{sep}} \to \mathrm{KK}_{\mathrm{sep}}$ that is (homotopy invariant) stable split-exact to an additive category. The target category $\mathrm{KK}_{\mathrm{sep}}$ is called the *Kasparov category*.

There are three main pictures to describe the cycles for KK-groups.

- Kasparov picture: homotopy classes of Kasparove modules.
- Cuntz picture: homotopy classes of quasi-homomorphisms or Cuntz pairs.
- Baaj-Julg picture: homotopy classes of unbounded Kasparov modules.

Theorem ([MN06])

The Kasparov category $\mathrm{KK}_{\mathrm{sep}}$ *is triangulated.*

A triangulated category is an additive category equipped with a “suspension functor”. Most examples of triangulated categories are realized as the homotopy categories of stable ∞-categories. Is the same for the Kasparov category?

Localization at KK-equivalences

Theorem ([LN18], [Hig87], [Hig88])

The ordinary KK-theory of separable C-algebras is exhibited by the ordinary localization* $\mathrm{C^*Alg_{sep}} \to \mathrm{KK_{sep}}$ *at KK-equivalence.*

Definition ([LN18])

The ∞-categorical KK-theory of separable C*-algebras is defined by the Dwyer-Kan localization $\mathrm{C^*Alg_{sep}} \to \mathrm{KK_{sep,\infty}}$ at KK-equivalence.

Theorem ([LN18])

The infinity category $\mathrm{KK_{sep,\infty}}$ *is stable.*

It is easy to see that $\mathrm{KK_{sep}}$ is equivalent to the homotopy category of the stable ∞-category $\mathrm{KK_{sep,\infty}}$.

Ind-completion by separable-finitary functor

Definition ([Ska88])

$$KK(A,B) := \operatorname*{colim}_{A_0 \subset A \text{ separable}} KK(A_0, B)$$

Bott periodicity, six-term exact sequences remain valid.

Definition ([BEL21])

Note that the category of C*-algebras is $\aleph_1$-presentable by a $\aleph_1$-filtered essentially small category of separable C*-algebras. This induces the following functor.

$$\mathrm{C^*Alg} \to \mathrm{KK}_\infty := \mathrm{Ind}(\mathrm{KK}_{\mathrm{sep},\infty}). \qquad \text{(extension by filtered colimits)}$$

Theorem ([BEL21])

$\mathrm{C^*Alg_{sep}} \to \mathrm{KK}_{\mathrm{sep},\infty}$ *is the universal stable semi-exact functor to a stable ∞-category.*
$\mathrm{C^*Alg} \to \mathrm{KK}_\infty$ *is the universal stable semi-exact separable-finitary to a cocomplete stable ∞-category.*

Homotopy theoretic construction

In [Bun24], Bunke succeeded in construction of KK-theory without any lifting theorems like Kasparov's technical theorem or Thomsen's noncommutative Tietze extension.

Repeated Dwyer-Kan localizations and the Bott periodicity give

$$\mathrm{C^*Alg_{sep}} \xrightarrow{L_h} \mathrm{C^*Alg}_{\mathrm{sep},h} \xrightarrow{L_{st}} \mathrm{C^*Alg}_{\mathrm{sep},h,st} \xrightarrow{L_{se}} \mathrm{C^*Alg}_{\mathrm{sep},h,st,se} \xrightarrow{\Omega^2} \mathrm{KK}_{\mathrm{sep},\infty}$$

Also for E-theory

$$\mathrm{C^*Alg_{sep}} \xrightarrow{L_h} \mathrm{C^*Alg}_{\mathrm{sep},h} \xrightarrow{L_{st}} \mathrm{C^*Alg}_{\mathrm{sep},h,st} \xrightarrow{L_{e}} \mathrm{C^*Alg}_{\mathrm{sep},h,st,e} \xrightarrow{\Omega^2} \mathrm{E}_{\mathrm{sep},\infty}$$

Finally take the Ind-completion.

He also investigated the relations with other classical pictures of KK-theory.

Potential applications

- The UCT class can be defined as the smallest stable subcategory of KK_∞ which is closed under colimits and contains $\mathbb{C}$. However, countable sums are not preserved in non-separable KK-theory, so we cannot see UCT is closed under countable sums in this new ∞-categorical KK-theory. No progress is made on the UCT problem.
- There are some results on algebraic K-theory of C^*-algebras using KK_∞.
  I don't know any deeper details.

Personal questions from an analytic side.

- Any other such properties that cannot be extended to non-separable algebras?
- Equivariant KK-theory over uncountable or topological groups?
- KK-cycles for non-separable algebras without filtered colimits?

References I

[BEL21] Ulrich Bunke, Alexander Engel, and Markus Land. A stable ∞-category for equivariant kk-theory, 2021. arXiv:2102.13372.

[Bun24] Ulrich Bunke. KK- and E-theory via homotopy theory. *Orbita Math.*, 1(2):103–210, 2024.

[Hig87] Nigel Higson. A characterization of KK-theory. *Pacific J. Math.*, 126(2):253–276, 1987.

[Hig88] Nigel Higson. Algebraic K-theory of stable C^*-algebras. *Adv. in Math.*, 67(1):140, 1988.

[Hin16] Vladimir Hinich. Dwyer-Kan localization revisited. *Homology Homotopy Appl.*, 18(1):27–48, 2016.

[LN18] Markus Land and Thomas Nikolaus. On the relation between K- and L-theory of C^*-algebras. *Math. Ann.*, 371(1-2):517–563, 2018.

[MN06] Ralf Meyer and Ryszard Nest. The Baum-Connes conjecture via localisation of categories. *Topology*, 45(2):209–259, 2006.

[Ska88] Georges Skandalis. Une notion de nucléarité en K-théorie (d'après J. Cuntz). *K-Theory*, 1(6):549–573, 1988.