A solution to Haagerup's problem on normal weights

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Kyoto, September 2025

Main result

In my master's thesis, I solved a 50-year-old problem posed by Haagerup in his master's thesis and obtained the following theorem!

Theorem ([Cho25], arXiv:2501.16832)

Let A be a C^*-algebra, and F^* be a weakly* closed convex hereditary subset of A^{*+}. Then, for any $\omega' \in A^{*+} \setminus F^*$, there exists $a \in A^+$ such that

$$\omega'(a) > 1 \qquad \text{and} \qquad \omega(a) \leq 1 \qquad \text{for all } \omega \in F^*.$$

The original proof has been simplified thanks to N. Ozawa.
I also simplified the solution of the Dixmier problem ([Dix53]) by Haagerup.

Theorem ([Haa75])

For a subadditive weight φ on M, the followings are equivalent:

- *φ is σ-weakly lower semi-continuous.*
- *φ is a supremum of normal positive linear functionals.*

Contents

Definitions of weights

We will always denote C*-algebras and vN algebras by A and M respectively.

Definition (Weights and subadditive weights)

A *weight* on A is a homogeneous additive functional $\varphi : A^+ \to [0,\infty]$, i.e.

$$\varphi(tx) = t\varphi(x), \qquad \varphi(x+y) = \varphi(x) + \varphi(y), \qquad t \geq 0,\ x, y \in A^+.$$

A *subadditive weight* on A is a homogeneous subadditive functional $\varphi : A^+ \to [0,\infty]$, i.e.

$$\varphi(tx) = t\varphi(x), \qquad \varphi(x+y) \leq \varphi(x) + \varphi(y), \qquad t \geq 0,\ x, y \in A^+.$$

A subadditive (= s.a.) weight may not be a weight.

A positive linear functional is exactly a bounded (or finite) weight.

Properties of weights

Recall that a vN algebra is a C*-algebra M with a (unique) predual M_*.
The weak* topology on M is conventionally called the σ-weak topology.

Definition (Properties of weights)

For a (s.a.) weight φ on A, we say it is

(i) *faithful* if $\varphi(a) = 0$ implies $a = 0$ for $a \in A^+$,

(ii) *densely defined* if $\varphi^{-1}([0,\infty))$ is norm dense in A^+,

(iii) *lower semi-continuous* if $\varphi^{-1}([0,1])$ is norm closed in A^+.

For a (s.a.) weight φ on M, we say it is

(ii') *semi-finite* if $\varphi^{-1}([0,\infty))$ is σ-weakly dense in M^+,

(iii') *normal* if $\varphi^{-1}([0,1])$ is σ-weakly closed in M^+.

Normality can be regarded as the generalization of the countable additivity of measures, and the σ-weak lower semi-continuity can be understood as the Fatou lemma.
"norm dense" and "norm closed" can be replaced into "weakly dense" and "weakly closed".

Motivating examples for weights

Example (Localizable measures)

A localizable measure μ is always a f.s.n. weight on $L^\infty(\mu)$.

Every M admits a f.s.n. weight.

(A localizable measure space is intuitively a presentation of commutative vN algebra.)

Example (Radon measures)

For a locally compact Hausdorff X, there are natural 1-1 correspondences among

- d.l. weights on $C_0(X)$,
- positive linear functionals on $C_c(X)$,
- locally finite inner regular Borel measures on X.

There is A without a f.d.l. weight.

(Every densely defined s.a. weight on a unital A is bounded.)

Gelfand-Naimark-Segal representations

Since a bounded weight $\omega \in A^{*+}$ defines a sesqui-linear form on A, by "separation and completion" we obtain a Hilbert space H, as in the contruction of L^2.
Then, we naturally have $\pi : A \to B(H)$ with a canonical vector $\Omega \in H$ such that $\overline{A\Omega} = H$.

Example

For a finite Radon measure $\mu \in C_0(X)^{*+}$, the associated GNS representation is the multiplication $C_0(X) \to B(L^2(\mu))$, and the cyclic vector Ω is the constant unit function.

Weights smoothly generalize this construction to the "unbounded measures".

Example (Semi-cyclic representations)

A *semi-cyclic representations* of A is a representation $\pi : A \to B(H)$ with a partially defined left A-linear map $\Lambda : A \to H$ of dense range.
There is a 1-1 correspondence between weights on A and unitary equivalence classes of semi-cyclic representations of A. (If a weight is bounded, $\Lambda(a) := a\Omega$.)

Equivalent characterizations for normality of weights

Theorem ([Haa75], Dixmier's problem on normal weights)

For a weight φ on M, the followings are all equivalent. (4)⇒(3)⇒(2)⇒(1) *are clear.*

(1) φ *is completely additive for positive elements;*

$$\varphi(\sum_i x_i) = \sum_i \varphi(x_i), \qquad x_i \in M^+.$$

(2) φ *preserves directed suprema;*

$$\varphi(\sup_i x_i) = \sup_i \varphi(x_i), \qquad x_i \uparrow \sup_i x_i \text{ in } M^+.$$

(3) φ *is σ-weakly lower semi-continuous;*

$$\varphi(\lim_i x_i) \le \liminf_i \varphi(x_i), \qquad x_i \to \lim_i x_i \ \sigma\text{-weakly in } M^+.$$

(4) φ *is a supremum of normal positive linear functionals;*

$$\varphi(x) = \sup_{\omega \le \varphi,\ \omega \in M_*^+} \omega(x), \qquad x \in M^+.$$

Contents

First problem

Problem (1.10 in [Haa75])

For a s.a. weight on M, if it preserves directed suprema (2)*, then is it σ-weakly lower semi-continuous* (3)*?*

Haagerup proved (1)⇒(3) for weights in the first half of [Haa75].
He proved first for σ-finite M, and extended to general M.
(M is called *σ-finite* or *countably decomposable* if it admits a faithful normal state.)

Theorem ([Haa75])

For a s.a. weight on general M, (1)⇒(3) *holds if*
(1)⇒(3) *holds on σ-finite vN subalgebras.*
For a weight on σ-finite M, (1)⇒(3) *holds.*

Thus, it is enough to solve the problem in the σ-finite case.

Second problem

Problem (1.11 in [Haa75])

For a weight on M, is it normal if it is normal on every commutative vN subalgebra?

Theorem ([Dix53])

For $\omega \in M^{*+}$, the followings are all equivalent.

(-1) *ω is completely additive for orthogonal projections.*

(0) *ω is completely additive for positive elements on commutative vN subalgebras.*

(1) *ω is completely additive for positive elements.*

(2) *ω preserves directed suprema.*

(3) *ω is σ-weakly continuous.*

(3)⇒(2)⇒(1)⇒(0)⇒(-1) are clear also for unbounded weights.

(-1)⇒(0) is false for unbounded weights. The question asks if (0)⇒(1) holds.

Third problem

Problem (2.7 in [Haa75])

Does the positive bipolar theorem hold for dual C^*-algerbas? See (d) *in the next slide.*

This problem is related to the proof of (3)⇒(4). We need some definitions to discuss.

Definition (Hereditary subsets and positive polars)

Let (E, E^*) be a dual pair of (directed partially) ordered real vector spaces such that E^+ and E^{*+} are mutually dual cones, i.e. $E^{*+} = \{x^* \in E : x^*(x) \geq 0 \text{ for } x \in E\}$.
For $F \subset E^+$, we say it is *hereditary* if $0 \leq x \leq y \in F$ implies $x \in F$,
and its *positive polar* is the positive part of the real polar

$$F^{r+} := (F^r)^+ = \{x^* \in E^* : \sup_{x \in F} x^*(x) \leq 1\}^+ = \{x^* \in E^{*+} : \sup_{x \in F} x^*(x) \leq 1\}.$$

Example (Hereditary C*-subalgebras)

A C*-subalgebra B of A is called *hereditary* if B^+ is hereditary in A^+.

Third problem

We focus on the real dual pairs (A^{sa}, A^{*sa}) and (M^{sa}, M_*^{sa}) with weak/weak* topologies.

Theorem ((a)∼(c) in [Haa75], (d) in [Cho25])

Consider the ordered real dual pairs (M^{sa}, M_*^{sa}) and (A^{sa}, A^{*sa}) of self-adjoint parts.

(a) *If F is a σ-weakly closed convex hereditary subset of M^+, then $F = F^{r+r+}$.*

(b) *If F_* is a norm closed convex hereditary subset of M_*^+, then $F_* = F_*^{r+r+}$.*

(c) *If F is a norm closed convex hereditary subset of A^+, then $F = F^{r+r+}$.*

(d) ***If F^* is weakly* closed convex hereditary subset of A^{*+}, then $F^* = (F^*)^{r+r+}$.***

They can be written in the form of Hahn-Banach separation, as written at the beginning.

I proved (d) and simplified the proofs of (a)∼(c).

Corollaries of third problem

Corollary ([Haa75])

For a s.a. weight φ on M, (3)⇒(4) holds.

(3) *φ is σ-weakly lower semi-continuous.*

(4) *φ is a supremum of normal positive linear functionals.*

Proof. Let

$$F := \{x \in M^+ : \varphi(x) \le 1\}, \qquad F_* := \{\omega \in M_*^+ : \omega \le \varphi\}.$$

Then, $F_* = F^{r+}$ by definition, and (4) is equivalent to $F = {F_*}^{r+}$.

F is convex and hereditary by definition of s.a. weights, and σ-weakly closed by (3).

So we are done by (a). □

Using (c) instead of (a), we can simplify the proof of the following theorem.

Corollary ([Com68])

A lower semi-continuous s.a. weight on A is a supremum of positive linear functionals.

Corollaries of third problem

Corollary ([Cho25])

There are 1-1 correspondences

$$\left\{\begin{array}{c}\textit{normal}\\ \textit{subadditive}\\ \textit{weights on } M\end{array}\right\} \longleftrightarrow \left\{\begin{array}{c}\sigma\textit{-weakly closed}\\ \textit{convex hereditary}\\ \textit{subsets of } M^+\end{array}\right\} \longleftrightarrow \left\{\begin{array}{c}\textit{norm closed}\\ \textit{convex hereditary}\\ \textit{subsets of } M_*^+\end{array}\right\}$$

and

$$\left\{\begin{array}{c}\textit{lower semi-}\\ \textit{continuous}\\ \textit{subadditive}\\ \textit{weights on } A\end{array}\right\} \longleftrightarrow \left\{\begin{array}{c}\textit{norm closed}\\ \textit{convex hereditary}\\ \textit{subsets of } A^+\end{array}\right\} \overset{(d)}{\longleftrightarrow} \left\{\begin{array}{c}\textit{weakly}^* \textit{ closed}\\ \textit{convex hereditary}\\ \textit{subsets of } A^{*+}\end{array}\right\}$$

Contents

Idea of (a)

To motivate the ideas, we sketch (a) and (b). Here are some preparations for (a).

Definition (Suppression by the one-parameter family of functional calculi)

For $\delta > 0$, we define a function $f_\delta : (-\delta^{-1}, \infty) \to \mathbb{R}$ such that

$$f_\delta(t) := t(1+\delta t)^{-1}, \qquad t > -\delta^{-1}.$$

Its graph is a concave hyperbola which approaches to the identity as $\delta \to 0$.
They are operator monotone, σ-strongly continuous, and has the semi-group property.

The domain issue $t > \delta^{-1}$ is highly critical.

Theorem (Krein-Šmulian theorem)

Let E be a Banach space, and let F^* be a convex subset of E^*.
Then, F^* is weakly closed if $F^* \cap E_r^*$ is weakly* closed for all $r > 0$, where*

$$E_r^* := \{x^* \in E^* : \|x^*\| \leq r\}.$$

Idea of (a)

Proof sketch of (a) by Haagerup. Since

$$F^{r+} = F^r \cap M_*^+ = F^r \cap (-M^+)^r = (F \cup -M^+)^r = (F - M^+)^r,$$
$$F^{r+r+} = (F - M^+)^{rr+} = (\overline{F - M^+})^+$$

by the usual real bipolar theorem, it suffices to solve the inclusion problem $(\overline{F - M^+})^+ \subset F$.

Heuristics. Let $x \in (\overline{F - M^+})^+$. Then, we have nets $x_i, y_i \in M^{sa}$ such that

$$x_i \to x \ \sigma\text{-weakly in } M, \qquad x_i \le y_i \in F.$$

If x_i were bounded by $r > 0$, then for each $0 < \delta < r^{-1}$ we can define $f_\delta(x_i)$.
Then, (assuming $x_i \to x$ σ-strongly by the Mazur and $f_\delta(y_i) \to y_\delta$ σ-weakly by the Alaoglu) we get from $x_i \le y_i \in F$

$$f_\delta(x_i) \le f_\delta(y_i) \in F \quad \Rightarrow \quad 0 \le f_\delta(x) \le f_\delta(y) \le y_\delta \in F \quad \Rightarrow \quad f_\delta(x) \in F \quad \Rightarrow \quad x \in F.$$

Idea of (a)

Question. How can we remove the boundedness assumption of x_i?

Solution. Use the Krein-Šmulian theorem. Define

$$G := \{x \in M^{sa} : \text{for any sufficiently small } \delta > 0,\ f_\delta(x) \in F - M^+\}.$$

It is enough to show

$$F - M^+ \subset G, \qquad G^+ \subset F, \qquad \overline{G} \subset G,$$

and the first two are clear.

The weak* closedness of G can be shown by the Krein-Šmulian theorem. □

Idea of (b)

To consider functional calculi of linear functionals, we introduce the following.
It transforms linear functionals to operators in the commutant for suitble representations.

Definition (Bounded commutant Radon-Nikodym derivatives)

Let $(\pi : M \to B(H),\ \Omega \in H)$ be the GNS representation of $\psi \in M_*^+$.
Then, there is a positive bounded linear map $\theta = \theta_\psi : \pi(M)' \to M_*$ defined such that

$$\theta(h)(x) := \langle h\pi(x)\Omega, \Omega\rangle, \qquad h \in \pi(M)',\ x \in M.$$

We will call this the *RN map* of ψ. (It is not a standard terminology.)

If $\omega \in M_*$ is dominated by ψ in the sense that $|\omega(x)| \le \psi(x)$ for all $x \in M^+$,
then ω is in the image of θ and $\theta^{-1}(\omega)$ is uniquely defined with $\|\theta^{-1}(\omega)\| \le 1$.
In this case, we have $\theta^{-1}(\omega) = d\omega/d\psi$ when M is commutative.

Idea of (b)

Proof sketch of (b). It is enough to prove $(\overline{F_* - M_*^+})^+ \subset F_*$. Let $\omega \in (\overline{F_* - M_*^+})^+$ with sequences $\omega_n, \varphi_n \in M_*^{sa}$ such that

$$\omega_n \to \omega \text{ in norm of } M_*, \qquad \omega_n \le \varphi_n \in F_*.$$

We may assume $\|\omega_n - \omega\| \le 2^{-n}$ for all n by passing to a subsequence. Define

$$\psi := \omega + \sum_n [\omega_n - \omega] + \sum_n 2^{-n} \frac{\varphi_n}{1 + \|\varphi_n\|} \in M_*^+.$$

Let $\theta : \pi(M)' \to M_*$ be the RN map associated to ψ. Then, we can define

$$\omega_\delta := \theta(f_\delta(\theta^{-1}(\omega))), \qquad \omega_{n,\delta} := \theta(f_\delta(\theta^{-1}(\omega_n))), \qquad \varphi_{n,\delta} := \theta(f_\delta(\theta^{-1}(\varphi_n)))$$

and prove $\omega \in F_*$ as in the proof of (a). □

Strategies for (d)

To prove (d), let $\omega \in (\overline{F^* - A^{*+}})^+$, and take nets $\omega_i, \varphi_i \in A^{*sa}$ such that

$$\omega_i \to \omega \text{ weakly}^* \text{ in } A^*, \qquad \omega_i \le \varphi_i \in F^*.$$

Question 1. How can we choose the reference ψ for the Radon-Nikodym?
Solution 1. Take ψ_i dynamically depending on ω_i.

Question 2. How can we commute the weak* limit of ω_i and f_δ without strong topology?
Solution 2. Approximate f_δ with affine functions by

$$t - \delta^{\frac{1}{2}} \le f_\delta(t) \le t, \qquad |t| \le 2^{-1}\delta^{-\frac{1}{4}},$$

$$(1+\delta^{-1})t \le f_\delta(t) \le t, \qquad 0 \le t \le 1.$$

These ideas can be also used to simplify the proof of (a).

Contents

Proof of (d)

Proof of (d). It suffices to show $(\overline{F^* - A^{*+}})^+ \subset F^*$. Define

$$G^* := \left\{ \omega \in A^{*sa} : \begin{array}{c} \text{there is } \psi \in A^{*+}, \text{ and there is } \varphi_\delta \in F^* \\ \text{for any sufficiently small } \delta > 0, \text{ such that} \\ \|\psi\| \le 1,\ \|\varphi_\delta\| \le \delta^{-1}, \text{ and } \omega \le \varphi_\delta + \delta^{\frac{1}{2}}\psi \end{array} \right\}.$$

It suffices to show $F^* - A^{*+} \subset G^*$, $G^{*+} \subset F^*$, and $\overline{G^*} \subset G^*$.

Proof of (d)

Step 1. Let $\omega \in F^* - A^{*+}$. Take $\varphi \in F^*$ such that $\omega \le \varphi$. Define, for $\delta > 0$,

$$\psi := \frac{[\omega]}{1+\|\omega\|} + \frac{\varphi}{(1+\|\omega\|)(1+\|\varphi\|)}, \qquad \varphi_\delta := \theta(f_\delta(\theta^{-1}(\varphi))),$$

where θ is the RN map associated to ψ. The norm conditions $\|\psi\| \le 1$ and $\|\varphi_\delta\| \le \delta^{-1}$ are easily checked. For sufficiently small $\delta > 0$ such that $\|\theta^{-1}(\omega)\| \le 1 + \|\omega\| \le 2^{-1}\delta^{-\frac{1}{4}}$ and $\delta \le 1$, we have

$$\theta^{-1}(\omega) \le f_\delta(\theta^{-1}(\omega)) + \delta^{\frac{1}{2}} \le f_\delta(\theta^{-1}(\varphi)) + \delta^{\frac{1}{2}},$$

so $\omega \le \varphi_\delta + \delta^{\frac{1}{2}}\psi$ and $\omega \in G^*$.

Proof of (d)

Step 2. Let $\omega \in G^{*+}$. Take $\psi \in A^{*+}$ and $\varphi_\delta \in F^*$ such that $\|\psi\| \le 1$, $\|\varphi_\delta\| \le \delta^{-1}$, $\omega \le \varphi_\delta + \delta^{\frac{1}{2}}\psi$, for any sufficiently small $\delta > 0$. Let $\psi_\delta := \omega + \delta\varphi + \psi$, and let θ_δ be the associated RN map. For any fixed $\delta' > 0$, since $0 \le \theta_\delta^{-1}(\omega) \le 1$, we have

$$
\begin{aligned}
0 &\le (1+\delta')^{-1}\theta_\delta^{-1}(\omega) \le f_{\delta'}(\theta_\delta^{-1}(\omega)) \le f_{\delta'}(\theta_\delta^{-1}(\varphi_\delta + \delta^{\frac{1}{2}}\psi)) \\
&\le f_{\delta'}(\theta_\delta^{-1}(\varphi_\delta) + \delta^{\frac{1}{2}}) \le f_{\delta'}(\theta_\delta^{-1}(\varphi_\delta)) + \delta^{\frac{1}{2}},
\end{aligned}
$$

and it implies

$$
0 \le (1+\delta')^{-1}\omega \le \theta_\delta(f_{\delta'}(\theta_\delta^{-1}(\varphi_\delta))) + \delta^{\frac{1}{2}}\psi_\delta.
$$

Since $\|\psi_\delta\| \le \|\omega\| + 2$ is bounded and $\theta_\delta(f_{\delta'}(\theta_\delta^{-1}(\varphi_\delta))) \in F^*$ is also bounded for fixed δ' as $\delta \to 0$, by considering the limit along a cofinal ultrafilter on the set of δ, we have $(1+\delta')^{-1}\omega \in F^*$, so $\delta' \to 0$ gives $\omega \in F^*$.

Proof of (d)

Step 3. To show G^* is weakly* closed, we claim for any $r > 0$ that

$$\overline{(F^* - A^{*+}) \cap A^*_{2r}} \subset G^*, \qquad G^* \cap A^*_r \subset \overline{(F^* - A^{*+}) \cap A^*_{2r}},$$

where $A^*_r := \{\omega \in A^* : \|\omega\| \le r\}$. If these are true, then

$$G^* \cap A^*_r = \overline{(F^* - A^{*+}) \cap A^*_{2r}} \cap A^*_r$$

is weakly* closed and convex in A^* for all $r > 0$, so the Krein-Šmulian theorem shows the claim.

Proof of (d)

Let $\omega_i \in (F^* - A^{*+}) \cap A^*_{2r}$ be a net such that $\omega_i \to \omega$ weakly* in A^*. Following the proof of $F^* - A^{*+} \subset G^*$, we can take $\psi_i \in A^{*+}$ and $\varphi_{i,\delta} \in F^*$ such that $\|\psi_i\| \le 1$, $\|\varphi_{i,\delta}\| \le \delta^{-1}$, $\omega_i \le \varphi_{i,\delta} + \delta^{\frac{1}{2}} \psi_i$, for uniformly sufficiently small δ such that $1 + 2r \le 2^{-1}\delta^{-\frac{1}{4}}$ because $\|\omega_i\|$ is bounded by $2r$. Since the three conditions are preserved by the weak* convergence, taking the limit along a cofinal ultrafilter on the index set of i, we can obtain limit points ψ and φ_δ so that $\omega \in G^*$.

Let $\omega \in G^* \cap A^*_r$. Take $\psi \in A^{*+}$ and $\varphi_\delta \in F^*$ with $\|\psi\| \le 1$, $\|\varphi_\delta\| \le \delta^{-1}$, $\omega \le \varphi_\delta + \delta^{\frac{1}{2}}\psi$, for any sufficiently small $\delta > 0$. If $\delta^{\frac{1}{2}} < r$, then $\omega - \delta^{\frac{1}{2}}\psi \in (F^* - A^{*+}) \cap A^*_{2r}$ converges to ω weakly* in A^* as $\delta \to 0$, we have $\omega \in \overline{(F^* - A^{*+}) \cap A^*_{2r}}$. □

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