

A solution to Haagerup's problem on normal weights

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Main result

In my master's thesis, I solved a 50-year-old problem posed by Haagerup in his master's thesis and obtained the following theorem!

Theorem ([Cho26])

Let A be a C^* -algebra, and F^* be a weakly* closed convex hereditary subset of A^{*+} . Then, for any $\omega' \in A^{*+} \setminus F^*$, there exists $a \in A^+$ such that

$$\omega'(a) > 1 \quad \text{and} \quad \omega(a) \leq 1 \quad \text{for all } \omega \in F^*.$$

I also simplified the solution of the Dixmier problem ([Dix53]) by Haagerup.

Theorem ([Haa75])

For a subadditive weight φ on M , the followings are equivalent:

- ▶ φ is σ -weakly lower semi-continuous.
- ▶ φ is a supremum of normal positive linear functionals.

Acknowledgement: my original proof has been simplified thanks to N. Ozawa.

Contents

1. Weights

2. Haagerup's three problems

3. Strategies

4. Proof

5. Applications

Definitions of weights

We will always denote C^* -algebras and vN algebras by A and M respectively.

Definition (Weights and subadditive weights)

A *weight* on A is a homogeneous additive functional $\varphi : A^+ \rightarrow [0, \infty]$, i.e.

$$\varphi(tx) = t\varphi(x), \quad \varphi(x+y) = \varphi(x) + \varphi(y), \quad t \geq 0, x, y \in A^+.$$

A *subadditive weight* on A is a homogeneous subadditive functional $\varphi : A^+ \rightarrow [0, \infty]$, i.e.

$$\varphi(tx) = t\varphi(x), \quad \varphi(x+y) \leq \varphi(x) + \varphi(y), \quad t \geq 0, x, y \in A^+.$$

A subadditive (= s.a.) weight may not be a weight.

A positive linear functional is exactly a bounded (or finite) weight.

Properties of weights

Recall that a vN algebra is a C^* -algebra M with a (unique) predual M_* .

The weak* topology on M is conventionally called the σ -weak topology.

Definition (Properties of weights)

For a (s.a.) weight φ on A , we say it is

- (i) *faithful* if $\varphi(a) = 0$ implies $a = 0$ for $a \in A^+$,
- (ii) *densely defined* if $\varphi^{-1}([0, \infty))$ is norm dense in A^+ ,
- (iii) *lower semi-continuous* if $\varphi^{-1}([0, 1])$ is norm closed in A^+ .

For a (s.a.) weight φ on M , we say it is

- (ii') *semi-finite* if $\varphi^{-1}([0, \infty))$ is σ -weakly dense in M^+ ,
- (iii') *normal* if $\varphi^{-1}([0, 1])$ is σ -weakly closed in M^+ .

Normality can be regarded as the generalization of the countable additivity of measures, as the σ -weak lower semi-continuity can be understood as the Fatou lemma.

“norm dense” and “norm closed” can be replaced into “weakly dense” and “weakly closed”.

Motivating examples for weights

Example (Localizable measures)

A localizable measure μ is always a f.s.n. weight on $L^\infty(\mu)$. Every M has a f.s.n. weight. One can say localizable measure spaces are concrete models for commutative vN algebras.

Example (Radon measures)

For a locally compact Hausdorff X , there are natural 1-1 correspondences among

- ▶ d.l. weights on $C_0(X)$,
- ▶ positive linear functionals on $C_c(X)$,
- ▶ locally finite inner regular Borel measures on X .

There is A without a f.d.l. weight.

(Every densely defined s.a. weight on a unital A is bounded.)

Gelfand-Naimark-Segal representations

Since a bounded weight $\omega \in A^{*+}$ defines a sesqui-linear form on A , by “separation and completion” we obtain a Hilbert space H , as in the construction of L^2 .

Then, we naturally have $\pi : A \rightarrow B(H)$ with a canonical vector $\Omega \in H$ such that $\overline{A\Omega} = H$.

Example

For a finite Radon measure $\mu \in C_0(X)^{*+}$, the associated GNS representation is the multiplication $C_0(X) \rightarrow B(L^2(\mu))$, and the cyclic vector Ω is the constant unit function.

We can generalize this construction to the weights.

Example (Semi-cyclic representations)

A *semi-cyclic representations* of A is a representation $\pi : A \rightarrow B(H)$ with a partially defined left A -linear map $\Lambda : \text{dom } \Lambda \subset A \rightarrow H$ of dense range.

There is a 1-1 correspondence between weights on M and unitary equivalence classes of semi-cyclic representations of M . (If a weight is bounded, $\Lambda(x) := x\Omega$.)

Equivalent characterizations for normality of weights

Theorem ([Haa75], Dixmier's problem on normal weights)

For a weight φ on M , the followings are all equivalent. (4) \Rightarrow (3) \Rightarrow (2) \Rightarrow (1) are clear.

(1) φ is completely additive for positive elements;

$$\varphi\left(\sum_i x_i\right) = \sum_i \varphi(x_i), \quad x_i \in M^+.$$

(2) φ preserves directed suprema;

$$\varphi\left(\sup_i x_i\right) = \sup_i \varphi(x_i), \quad x_i \uparrow \sup_i x_i \text{ in } M^+.$$

(3) φ is σ -weakly lower semi-continuous;

$$\varphi\left(\lim_i x_i\right) \leq \liminf_i \varphi(x_i), \quad x_i \rightarrow \lim_i x_i \text{ } \sigma\text{-weakly in } M^+.$$

(4) φ is a supremum of normal positive linear functionals;

$$\varphi(x) = \sup_{\omega \leq \varphi, \omega \in M_*^+} \omega(x), \quad x \in M^+.$$

Contents

1. Weights

2. Haagerup's three problems

3. Strategies

4. Proof

5. Applications

First problem

Problem (1.10 in [Haa75])

For a s.a. weight φ on M , does (2) \Rightarrow (3) hold?

(2) φ preserves directed suprema.

(3) φ is σ -weakly lower semi-continuous.

Haagerup proved (1) \Rightarrow (3) for weights in the first half of [Haa75].

He proved first for σ -finite M , and extended to general M .

(M is called σ -finite or *countably decomposable* if it admits a faithful normal state.)

Theorem ([Haa75])

For a s.a. weight on M , if (1) \Rightarrow (3) holds on σ -finite vN subalgebras, then (1) \Rightarrow (3) holds.

For a weight on σ -finite M , (1) \Rightarrow (3) holds.

Thus, it is enough to solve the problem in the σ -finite case.

Second problem

Problem (1.11 in [Haa75])

For a weight on M , is it normal if it is normal on every commutative vN subalgebra?

Theorem ([Dix53])

For $\omega \in M^{+}$, the followings are all equivalent.*

- (-1) ω is completely additive for orthogonal projections.*
- (0) ω is completely additive for positive elements on commutative vN subalgebras.*
- (1) ω is completely additive for positive elements.*
- (2) ω preserves directed suprema.*
- (3) ω is σ -weakly continuous.*

$(3) \Rightarrow (2) \Rightarrow (1) \Rightarrow (0) \Rightarrow (-1)$ are clear also for unbounded weights.

$(-1) \Rightarrow (0)$ is false for unbounded weights. The question asks if $(0) \Rightarrow (1)$ holds.

Third problem

Problem (2.7 in [Haa75])

Does the positive bipolar theorem hold for dual C^* -algebras? See (d) in the next slide.

This problem is related to the proof of (3) \Rightarrow (4). We need some definitions for discussion.

Definition (Hereditary subsets and positive polars)

Let (E, E^*) be a dual pair of (directed partially) ordered real vector spaces such that E^+ and E^{*+} are mutually dual cones, i.e. $E^{*+} = \{x^* \in E^* : x^*(x) \geq 0 \text{ for } x \in E\}$.

For $F \subset E^+$, we say it is *hereditary* if $0 \leq x \leq y \in F$ implies $x \in F$, and its *positive polar* is the positive part of the real polar

$$F^{r+} := (F^r)^+ = \{x^* \in E^* : \sup_{x \in F} x^*(x) \leq 1\}^+ = \{x^* \in E^{*+} : \sup_{x \in F} x^*(x) \leq 1\}.$$

Example (Hereditary C^* -subalgebras)

A C^* -subalgebra B of A is called *hereditary* if B^+ is hereditary in A^+ .

Third problem

We focus on the real dual pairs (A^{sa}, A^{*sa}) and (M^{sa}, M_*^{sa}) with weak/weak* topologies.

Theorem ((a)~(c) in [Haa75], (d) in [Cho26])

Consider the ordered real dual pairs (M^{sa}, M_*^{sa}) and (A^{sa}, A^{*sa}) of self-adjoint parts.

- (a) If F is a σ -weakly closed convex hereditary subset of M^+ , then $F = F^{r+r+}$.
- (b) If F_* is a norm closed convex hereditary subset of M_*^+ , then $F_* = F_*^{r+r+}$.
- (c) If F is a norm closed convex hereditary subset of A^+ , then $F = F^{r+r+}$.
- (d) If F^* is weakly* closed convex hereditary subset of A^{*+} , then $F^* = (F^*)^{r+r+}$.**

They can be written in the form of Hahn-Banach separation, as in the beginning.

I proved (d) and simplified all the proofs of (a)~(c) by Haagerup.

Corollaries of third problem

Corollary ([Haa75])

For a s.a. weight φ on M , (3) \Rightarrow (4) holds.

(3) φ is σ -weakly lower semi-continuous.

(4) φ is a supremum of normal positive linear functionals.

Proof. Let

$$F := \{x \in M^+ : \varphi(x) \leq 1\},$$

which is convex and hereditary by definition of φ , and σ -weakly closed by (3). Then,

$$F^{r+} = \{\omega \in M_*^+ : \omega \leq \varphi\},$$

and (4) is equivalent to $F = F^{r+r+}$. So we are done by (a). □

Using (c) instead of (a), we can give another proof of the following.

Corollary ([Com68])

A lower semi-continuous s.a. weight on A is a supremum of positive linear functionals.

Corollaries of third problem

Corollary ([Cho26])

There are 1-1 correspondences

$$\left\{ \begin{array}{l} \text{normal} \\ \text{subadditive} \\ \text{weights on } M \end{array} \right\} \leftrightarrow \left\{ \begin{array}{l} \sigma\text{-weakly closed} \\ \text{convex hereditary} \\ \text{subsets of } M^+ \end{array} \right\} \leftrightarrow \left\{ \begin{array}{l} \text{norm closed} \\ \text{convex hereditary} \\ \text{subsets of } M_*^+ \end{array} \right\}$$

and

$$\left\{ \begin{array}{l} \text{lower semi-} \\ \text{continuous} \\ \text{subadditive} \\ \text{weights on } A \end{array} \right\} \leftrightarrow \left\{ \begin{array}{l} \text{norm closed} \\ \text{convex hereditary} \\ \text{subsets of } A^+ \end{array} \right\} \stackrel{(d)}{\leftrightarrow} \left\{ \begin{array}{l} \text{weakly}^* \text{ closed} \\ \text{convex hereditary} \\ \text{subsets of } A^{*+} \end{array} \right\}$$

Contents

1. Weights

2. Haagerup's three problems

3. Strategies

4. Proof

5. Applications

Idea of (a)

To motivate the ideas, we sketch (a) and (b). Here are some preparations for (a).

Definition (Suppression by the one-parameter family of functional calculi)

For $\delta > 0$, we define a function $f_\delta : (-\delta^{-1}, \infty) \rightarrow \mathbb{R}$ such that

$$f_\delta(t) := t(1 + \delta t)^{-1}, \quad t > -\delta^{-1}.$$

Its graph is a concave hyperbola which approaches to the identity as $\delta \rightarrow 0$.

They are operator monotone, σ -strongly continuous, and has the semi-group property.

The lower bound of the domain $t > -\delta^{-1}$ is a highly critical issue.

Theorem (Krein-Šmulian theorem)

Let E be a Banach space, and let F^* be a convex subset of E^* .

Then, F^* is weakly* closed if $F^* \cap E_r^*$ is weakly* closed for all $r > 0$, where

$$E_r^* := \{x^* \in E^* : \|x^*\| \leq r\}.$$

Idea of (a)

Proof sketch of (a) by Haagerup. Since

$$F^{r+} = F^r \cap M_*^+ = F^r \cap (-M^+)^r = (F \cup -M^+)^r = (F - M^+)^r,$$

$$F^{r+r+} = (F - M^+)^{r+r+} = (\overline{F - M^+})^+$$

by the usual real bipolar theorem, it suffices to solve the inclusion problem $(\overline{F - M^+})^+ \subset F$.

Heuristics. Let $x \in (\overline{F - M^+})^+$. Then, we have nets $x_i, y_i \in M^{sa}$ such that

$$x_i \rightarrow x \text{ } \sigma\text{-weakly in } M, \quad x_i \leq y_i \in F.$$

If x_i were bounded by $r > 0$, then for each $0 < \delta < r^{-1}$ we can define $f_\delta(x_i)$.

Then, (assuming $x_i \rightarrow x$ σ -strongly by the Mazur and $f_\delta(y_i) \rightarrow y_\delta$ σ -weakly by the Alaoglu) we get from $x_i \leq y_i \in F$

$$f_\delta(x_i) \leq f_\delta(y_i) \in F \quad \Rightarrow \quad 0 \leq f_\delta(x) \leq f_\delta(y) \leq y_\delta \in F \quad \Rightarrow \quad f_\delta(x) \in F \quad \Rightarrow \quad x \in F.$$

Idea of (a)

Question. How can we assume the boundedness of x_i ?

Solution. Use the Krein-Šmulian theorem. Define

$$G := \{x \in M^{sa} : \text{for any sufficiently small } \delta > 0, f_\delta(x) \in F - M^+\}.$$

It is enough to show

$$F - M^+ \subset G, \quad G^+ \subset F, \quad \overline{G} \subset G,$$

and the first two are clear.

The weak* closedness of G can be shown by the Krein-Šmulian theorem. □

Idea of (b)

To consider functional calculi of linear functionals, we introduce the following. It transforms linear functionals to operators in the commutant for suitable representations.

Definition (Bounded commutant Radon-Nikodym derivatives)

Let $(\pi : M \rightarrow B(H), \Omega \in H)$ be the GNS representation of $\psi \in M_*^+$.

Then, there is a positive bounded linear map $\theta = \theta_\psi : \pi(M)' \rightarrow M_*$ defined such that

$$\theta(h)(x) := \langle h\pi(x)\Omega, \Omega \rangle, \quad h \in \pi(M)', \quad x \in M.$$

We will call this the *RN map* of ψ . (It is not a standard terminology.)

If $\omega \in M_*$ is dominated by ψ in the sense that $|\omega(x)| \leq \psi(x)$ for all $x \in M^+$, then ω is in the image of θ and $\theta^{-1}(\omega)$ is uniquely defined with $\|\theta^{-1}(\omega)\| \leq 1$.

In this case, we have $\theta^{-1}(\omega) = d\omega/d\psi$ when M is commutative.

Idea of (b)

Proof sketch of (b). It is enough to prove $(\overline{F_* - M_*^+})^+ \subset F_*$. Let $\omega \in (\overline{F_* - M_*^+})^+$ with sequences $\omega_n, \varphi_n \in M_*^{sa}$ such that

$$\omega_n \rightarrow \omega \text{ in norm of } M_*, \quad \omega_n \leq \varphi_n \in F_*.$$

We may assume $\|\omega_n - \omega\| \leq 2^{-n}$ for all n by passing to a subsequence. Define

$$\psi := \omega + \sum_n [\omega_n - \omega] + \sum_n 2^{-n} \frac{\varphi_n}{1 + \|\varphi_n\|} \in M_*^+.$$

Let $\theta : \pi(M)' \rightarrow M_*$ be the RN map associated to ψ . Then, we can define

$$\omega_\delta := \theta(f_\delta(\theta^{-1}(\omega))), \quad \omega_{n,\delta} := \theta(f_\delta(\theta^{-1}(\omega_n))), \quad \varphi_{n,\delta} := \theta(f_\delta(\theta^{-1}(\varphi_n)))$$

and prove $\omega \in F_*$ as in the proof of (a). □

Strategies for (d)

To prove (d), let $\omega \in \overline{(F^* - A^{**})}^+$, and take nets $\omega_i, \varphi_i \in A^{**}$ such that

$$\omega_i \rightarrow \omega \text{ weakly}^* \text{ in } A^*, \quad \omega_i \leq \varphi_i \in F^*.$$

Question 1. How can we choose the reference ψ for the Radon-Nikodym?

Solution 1. Take ψ_i dynamically depending on ω_i .

Question 2. How can we commute the weak* limit of ω_i and f_δ without strong topology?

Solution 2. Approximate f_δ with affine functions by

$$t - \delta^{\frac{1}{2}} \leq f_\delta(t) \leq t, \quad |t| \leq 2^{-1} \delta^{-\frac{1}{4}},$$

$$(1 + \delta^{-1})t \leq f_\delta(t) \leq t, \quad 0 \leq t \leq 1.$$

These ideas can be also used to simplify the proof of (a).

Contents

1. Weights

2. Haagerup's three problems

3. Strategies

4. Proof

5. Applications

Proof of (d)

Proof of (d). It suffices to show $(\overline{F^* - A^{*+}})^+ \subset F^*$. Define

$$G^* := \left\{ \begin{array}{l} \text{there is } \psi \in A^{*+}, \text{ and there is } \varphi_\delta \in F^* \\ \omega \in A^{*sa} : \text{ for any sufficiently small } \delta > 0, \text{ such that} \\ \|\psi\| \leq 1, \|\varphi_\delta\| \leq \delta^{-1}, \text{ and } \omega \leq \varphi_\delta + \delta^{\frac{1}{2}}\psi \end{array} \right\}.$$

It suffices to show $F^* - A^{*+} \subset G^*$, $G^{*+} \subset F^*$, and $\overline{G^*} \subset G^*$.

Proof of (d)

Step 1. Let $\omega \in F^* - A^{*+}$. Take $\varphi \in F^*$ such that $\omega \leq \varphi$. Define, for $\delta > 0$,

$$\psi := \frac{[\omega]}{1 + \|\omega\|} + \frac{\varphi}{(1 + \|\omega\|)(1 + \|\varphi\|)}, \quad \varphi_\delta := \theta(f_\delta(\theta^{-1}(\varphi))),$$

where θ is the RN map associated to ψ . The norm conditions $\|\psi\| \leq 1$ and $\|\varphi_\delta\| \leq \delta^{-1}$ are easily checked. For sufficiently small $\delta > 0$ such that $\|\theta^{-1}(\omega)\| \leq 1 + \|\omega\| \leq 2^{-1}\delta^{-\frac{1}{4}}$ and $\delta \leq 1$, we have

$$\theta^{-1}(\omega) \leq f_\delta(\theta^{-1}(\omega)) + \delta^{\frac{1}{2}} \leq f_\delta(\theta^{-1}(\varphi)) + \delta^{\frac{1}{2}},$$

so $\omega \leq \varphi_\delta + \delta^{\frac{1}{2}}\psi$ and $\omega \in G^*$.

Proof of (d)

Step 2. Let $\omega \in G^{**}$. Take $\psi \in A^{**}$ and $\varphi_\delta \in F^*$ such that $\|\psi\| \leq 1$, $\|\varphi_\delta\| \leq \delta^{-1}$, $\omega \leq \varphi_\delta + \delta^{\frac{1}{2}}\psi$, for any sufficiently small $\delta > 0$. Let $\psi_\delta := \omega + \delta\varphi + \psi$, and let θ_δ be the associated RN map. For any fixed $\delta' > 0$, since $0 \leq \theta_\delta^{-1}(\omega) \leq 1$, we have

$$\begin{aligned} 0 &\leq (1 + \delta')^{-1} \theta_\delta^{-1}(\omega) \leq f_{\delta'}(\theta_\delta^{-1}(\omega)) \leq f_{\delta'}(\theta_\delta^{-1}(\varphi_\delta + \delta^{\frac{1}{2}}\psi)) \\ &\leq f_{\delta'}(\theta_\delta^{-1}(\varphi_\delta) + \delta^{\frac{1}{2}}) \leq f_{\delta'}(\theta_\delta^{-1}(\varphi_\delta)) + \delta^{\frac{1}{2}}, \end{aligned}$$

and it implies

$$0 \leq (1 + \delta')^{-1} \omega \leq \theta_\delta(f_{\delta'}(\theta_\delta^{-1}(\varphi_\delta))) + \delta^{\frac{1}{2}}\psi_\delta.$$

Since $\|\psi_\delta\| \leq \|\omega\| + 2$ is bounded and $\theta_\delta(f_{\delta'}(\theta_\delta^{-1}(\varphi_\delta))) \in F^*$ is also bounded for fixed δ' as $\delta \rightarrow 0$, by considering the limit along a cofinal ultrafilter on the set of δ , we have $(1 + \delta')^{-1}\omega \in F^*$, so $\delta' \rightarrow 0$ gives $\omega \in F^*$.

Proof of (d)

Step 3. To show G^* is weakly* closed, we claim for any $r > 0$ that

$$\overline{(F^* - A^{*+}) \cap A_{2r}^*} \subset G^*, \quad G^* \cap A_r^* \subset \overline{(F^* - A^{*+}) \cap A_{2r}^*},$$

where $A_r^* := \{\omega \in A^* : \|\omega\| \leq r\}$. If these are true, then

$$G^* \cap A_r^* = \overline{(F^* - A^{*+}) \cap A_{2r}^*} \cap A_r^*$$

is weakly* closed and convex in A^* for all $r > 0$, so the Krein-Šmulian theorem shows the claim.

Proof of (d)

Let $\omega_i \in (F^* - A^{*+}) \cap A_{2r}^*$ be a net such that $\omega_i \rightarrow \omega$ weakly* in A^* . Following the proof of $F^* - A^{*+} \subset G^*$, we can take $\psi_i \in A^{*+}$ and $\varphi_{i,\delta} \in F^*$ such that $\|\psi_i\| \leq 1$, $\|\varphi_{i,\delta}\| \leq \delta^{-1}$, $\omega_i \leq \varphi_{i,\delta} + \delta^{\frac{1}{2}}\psi_i$, for uniformly sufficiently small δ such that $1 + 2r \leq 2^{-1}\delta^{-\frac{1}{4}}$ because $\|\omega_i\|$ is bounded by $2r$. Since the three conditions are preserved by the weak* convergence, taking the limit along a cofinal ultrafilter on the index set of i , we can obtain limit points ψ and φ_δ so that $\omega \in G^*$.

Let $\omega \in G^* \cap A_r^*$. Take $\psi \in A^{*+}$ and $\varphi_\delta \in F^*$ with $\|\psi\| \leq 1$, $\|\varphi_\delta\| \leq \delta^{-1}$, $\omega \leq \varphi_\delta + \delta^{\frac{1}{2}}\psi$, for any sufficiently small $\delta > 0$. If $\delta^{\frac{1}{2}} < r$, then $\omega - \delta^{\frac{1}{2}}\psi \in (F^* - A^{*+}) \cap A_{2r}^*$ converges to ω weakly* in A^* as $\delta \rightarrow 0$, we have $\omega \in \overline{(F^* - A^{*+}) \cap A_{2r}^*}$. \square

Contents

1. Weights

2. Haagerup's three problems

3. Strategies

4. Proof

5. Applications

Applications

No applications...

If you have a plan to write a textbook on von Neumann algebras, I would be very happy if you use my result...

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